The St Venant Equations

Dr P A Sleigh
Dr I M Goodwill

School of Civil Engineering,
University of Leeds
March 2000

1 The derivation of the continuity equation

Consider a short length, Δx, of channel

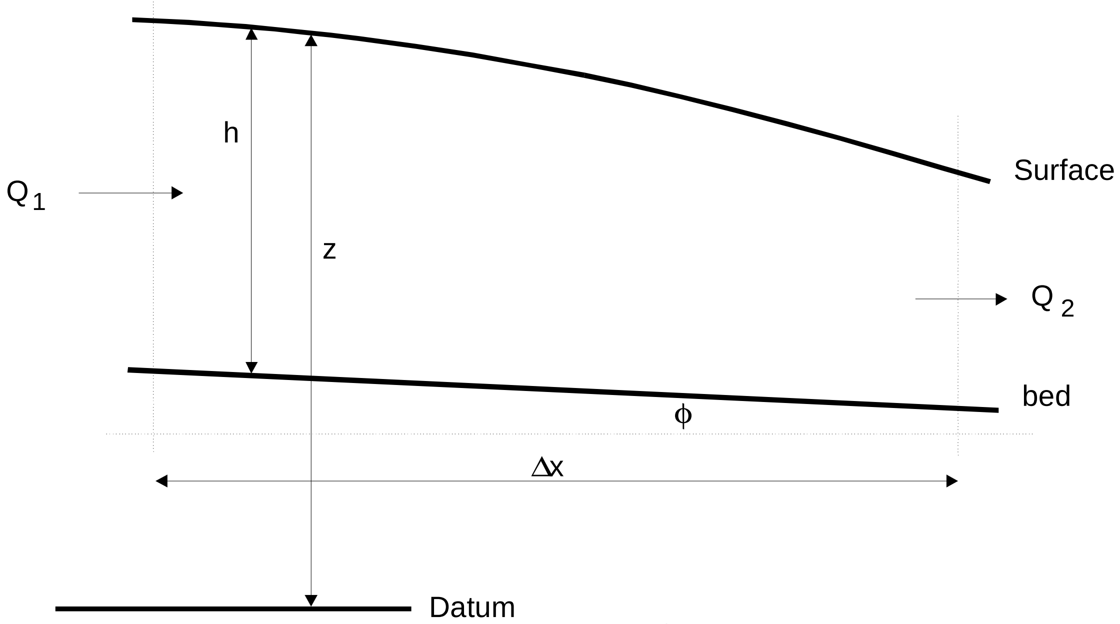

Figure 1 - Short length of channel

The following symbols are used in this derivation:

A = the cross-sectional area of the section
h = depth of flow at the section
z = elevation of surface above a datum at the section
v = mean velocity at the section
Q = discharge at the section
b = width of the top of the section
x = position of the section measured from the upstream end
t = time
g = acceleration due to gravity
ρ = mass density of the fluid

Others symbols are defined in the text at the point when they are introduced.

Assuming that there is no lateral inflow, then

$$Q_2 - Q_1 = \frac{\partial Q}{\partial x}\Delta x$$

This has the partial derivative since Q is changing with both x and time, t.

Now the volume of water between the sections 1 and 2 is increasing as a rate of

$$b\frac{\partial h}{\partial t}\Delta x$$

where b is the top width,

As cross-sectional area $A = bh$ then this is equivalent to

$$\frac{\partial A}{\partial t}\Delta x$$

The terms are equal in magnitude but of opposite sign, so

$$\frac{\partial Q}{\partial x}\Delta x + b\frac{\partial h}{\partial t}\Delta x = 0$$

as $\frac{\partial Q}{\partial x} = \frac{\partial (Av)}{\partial x}$ then

$$v\frac{\partial A}{\partial x} + A\frac{\partial v}{\partial x} + b\frac{\partial h}{\partial t} = 0 \qquad (1)$$

This is the **continuity equation**

2 The derivation of the dynamic or momentum equation.

By applying Newton's 2^{nd} law to our elemental length of channel we have

$$\begin{aligned} \text{Force} &= \text{mass} \times \text{acceleration} \\ &= \rho A \Delta x \frac{dv}{dt} \\ &= \rho A \Delta x \left[v\frac{\partial v}{\partial x} + \frac{\partial v}{\partial t} \right] \end{aligned}$$

since v varies with both space (x) and time (t)

Consider the external forces which cause this acceleration. These are, in the simplest case, three

$\frac{\partial H}{\partial x}$ change in static pressure

F frictional resistance of channel walls and bed

ρg gravity force (the weight)

If ϕ is the bed slope (measured positive as the bed rises from downstream to up – see Figure 1) then the sum of these three forces is

$$\frac{\partial H}{\partial x}\Delta x\cos\phi - F\Delta x + \rho gA\Delta x\sin\phi$$

For small bed slopes, ϕ, then $\cos\phi = 1$ and $\sin\phi = \phi = i$ so

$$\frac{\partial H}{\partial x}\Delta x - F\Delta x + \rho gA\Delta x i$$

Now

$$\frac{\partial H}{\partial x} = -\rho gA\frac{\partial h}{\partial x}$$

and

$$F = \rho gAj$$

where j is *energy loss / unit length of channel / unit weight of fluid*.

Equation these external forces to the change in momentum yields

$$\rho A\Delta x\left(v\frac{\partial v}{\partial x} + \frac{\partial v}{\partial t}\right) = -\rho gA\frac{\partial h}{\partial x}\Delta x - \rho gA\,j + \rho gA\,\Delta x\,i$$

Rearranging gives

$$g\frac{\partial h}{\partial x} + v\frac{\partial v}{\partial x} + \frac{\partial v}{\partial t} = g(i - j) \qquad (2)$$

This is the **dynamic**, or **momentum**, **equation**.

Using the Chezy expression, j, can be written

$$j = \frac{v^2}{C^2 m}$$

where C is the *ChezyC* and m is the hydraulic mean radius given by

$$m = \frac{\text{Area}}{\text{Wetted perimeter}} = \frac{A}{p}$$

3 The solution of the St Venant equations

The St Venant equations are:

The continuity equation

$$v\frac{\partial A}{\partial x}+A\frac{\partial v}{\partial x}+b\frac{\partial h}{\partial t}=0 \tag{1}$$

The dynamic, or momentum, equation

$$g\frac{\partial h}{\partial x}+v\frac{\partial v}{\partial x}+\frac{\partial v}{\partial t}=g(i-j) \tag{2}$$

The St Venant equations cannot be solved explicitly except by making some very large assumptions which are unrealistic for most situations. Therefore **numerical** techniques have to be used.

Equations 1 and 2 are not in a readily usable form to solve, so the first task is to rearrange them.

The problem is that *A* is a function of *h* so we need to re-write the equations using *h* and to arrange them so we can solve for the two unknowns *h* and *u*.

Assume the simplest channel that is, rectangular cross-section of constant slope.

3.1 An more convenient form of the equations

It is easier if we write the equations in terms of celerity, *c*, rather than *h* using this equation for celerity of a small gravity wave in a rectangular channel of depth *h*:

$$c=\sqrt{gh}$$
$$h=\frac{c^2}{g}$$

All the terms involving *h* of equations (1) and (2) can be replaced by terms containing *c* making use of

$$d(gh)=d\left(c^2\right)=2c\,dc$$
$$dh=\frac{2c}{g}dc$$

For the continuity equation (1), where *A* = *bh*, we get

$$bv\frac{\partial h}{\partial x}+bh\frac{\partial v}{\partial x}+b\frac{\partial h}{\partial t}=0$$
$$\frac{2c}{g}bv\frac{\partial c}{\partial x}+b\frac{c^2}{g}\frac{\partial v}{\partial x}+\frac{2c}{g}b\frac{\partial c}{\partial t}=0$$
$$2v\frac{\partial c}{\partial x}+c\frac{\partial v}{\partial x}+2\frac{\partial c}{\partial t}=0 \tag{3}$$

For the dynamic equation (2) we get

$$2c\frac{\partial c}{\partial x}+v\frac{\partial v}{\partial x}+\frac{\partial v}{\partial t}=g(i-j) \tag{4}$$

Adding equations (3) and (4) gives

$$\frac{\partial v}{\partial t}+(v+c)\frac{\partial v}{\partial x}+2\frac{\partial c}{\partial t}+2(v+c)\frac{\partial c}{\partial x}=g(i-j) \tag{5}$$

Subtracting equation (4) from (3) gives

$$\frac{\partial v}{\partial t}+(v-c)\frac{\partial v}{\partial x}-2\frac{\partial c}{\partial t}-2(v-c)\frac{\partial c}{\partial x}=g(i-j) \tag{6}$$

Equations (5) and (6) can be rearranged to give respectively

$$(v+c)\frac{\partial(v+2c)}{\partial x}+\frac{\partial(v+2c)}{\partial t}=g(i-j) \tag{7}$$

$$(v-c)\frac{\partial(v-2c)}{\partial x}+\frac{\partial(v-2c)}{\partial t}=g(i-j) \tag{8}$$

3.2 A diversion ….

In differential calculus, one of the basic equations of partial differentiation is

$$\frac{d\phi}{dt}=\frac{\partial\phi}{\partial x}\frac{dx}{dt}+\frac{\partial\phi}{\partial t}$$

where *x* and *t* are independent variables.

In open channel hydraulics we may think of this in a physical sense. If the variable ϕ is some property of the flow e.g. surface level, that varies with both distance (*x*) and time (*t*) then if an observer is moving at velocity $v=\frac{dx}{dt}$, the observer will see the surface level change only with time relative to the observers' position.

3.3 The characteristic form of the St Venant equations

As ϕ could be any variable, we can write it as $\phi = (v + 2c)$ then

$$\frac{d(v+2c)}{dt}=\frac{\partial(v+2c)}{\partial x}\frac{dx}{dt}+\frac{\partial(v+2c)}{\partial t}$$

Compare this equation with the left-hand side of equation (7)

They are equivalent if $\frac{dx}{dt} = (v + c)$, so equation (7) can be re-written as the total differential

$$\frac{d(v + 2c)}{dt} = g(i - j) \tag{9}$$

And similarly equation (8) can be re-written in total differential form if $\frac{dx}{dt} = (v - c)$ as

$$\frac{d(v - 2c)}{dt} = g(i - j) \tag{10}$$

Physically these can be interpreted as, for equation (9), if an observer is moving at velocity *(v+c)* then they see / experience changes given by (9).

And for equation (10), if the observer is moving at velocity *(v-c)* then they will see / experience changes given by (10).

These pairs are known as the **Characteristic form** of the St Venant Equations

$$\frac{d(v + 2c)}{dt} = g(i - j) \qquad \text{for} \qquad \frac{dx}{dt} = (v + c) \tag{11}$$

and

$$\frac{d(v - 2c)}{dt} = g(i - j) \qquad \text{for} \qquad \frac{dx}{dt} = (v - c) \tag{12}$$

You may also see them quoted as one pair using the ± symbol i.e.

$$\frac{d(v \pm 2c)}{dt} = g(i - j) \qquad \text{for} \qquad \frac{dx}{dt} = (v \pm c)$$

The solution of this system of equations is quite straight forward - particularly using a computer.

3.4 The meaning of the characteristic form

Consider the graph below which is in the *x* - *t* plane.

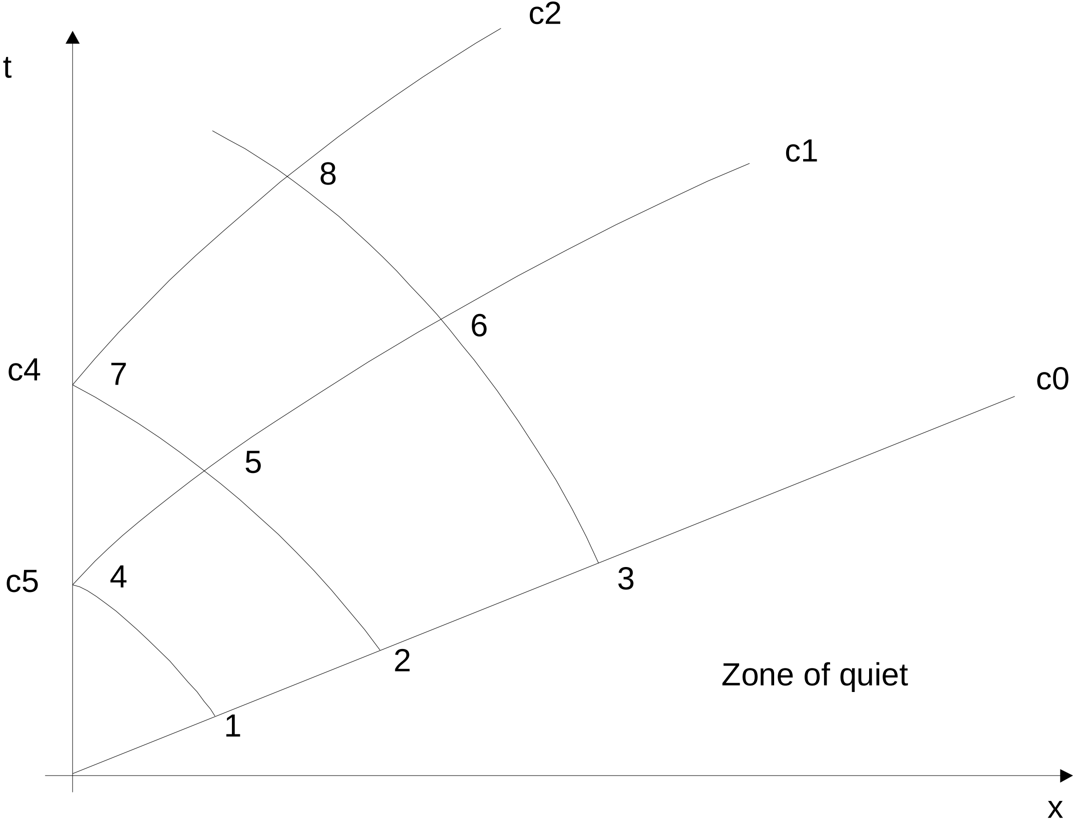

Figure 2, Characteristics on an x-t plane.

The paths of these observers that we have talked about can be represented by lines on this graph. That is to say, lines of gradient $\frac{dt}{dx}=\frac{1}{(v+c)}$ or $\frac{dt}{dx}=\frac{1}{(v-c)}$ on this graph are the paths of an observer. They may start at any point and are known as **characteristic lines** or simply **characteristics**. Along these lines the associated characteristic equation applies. It is possible to solve the characteristic equations at the point where they cross – at the same time this is the solution to the St Venant equations. This technique is known as the **Method of Characteristics**.

In practice the method of characteristics is becomes cumbersome to apply exactly on a computer, however there are many different ways that have proven successful, two of which will be shown below. Before that one very important results can be obtained from looking at the significance of the characteristics and their slopes – that of *numerical stability*.

Consider again the graph above and a disturbance at the upstream end of a channel (e.g. a flood wave). The part of the channel some way along from the end will not receive the disturbance for some time. The time it takes depends on the velocity that the information from the disturbance travels. In the graph of figure 1 the line C0 represents this velocity. Everything below that line represent the channel waiting for the disturbance – this is known as the **zone of quite**. C0 is a characteristic line. It is a **forward characteristic** as are line C1 and C2, they have gradient of the form $v+c$, (note though that each has a different gradient.) Lines C3, C4 and C5 are **negative (or backward) characteristics** and have gradients of the form v-c.
The values v and c at point 5 is influenced by events and conditions at points 0, 1, 2 and 4. But not by values at any other points – outside this region the values of v and c do not affect the values of v and c at point 5.

Similarly at point 2, say, the values of *v* and *c* will influence the values of *v* and *c* at points 5, 6 and 7 etc. but not at point 4.

3.5 An example computation using the characteristic equations

Assume we know the solution at points 3 and 5 and we want to determine the solution at point 6. We know that a forward characteristic through 5 intersects with a backward characteristic through 3. If the conditions at point 3 and 5 are respectively v_3, c_3 and v_5, c_5 then from (11) and (12) the forward characteristic from 5 starts off with gradient $\frac{1}{(v_5 + c_5)}$ and the negative characteristic from 3 starts off with gradient $\frac{1}{(v_3 - c_3)}$.

Along the forward characteristic applies

$$\frac{d(v+2c)}{dt} = g(i-j)$$

or

$$(v_6 + 2c_6) - (v_5 + 2c_5) = \Delta t\, g(i-j) \tag{13}$$

and along the backward characteristic

$$\frac{d(v-2c)}{dt} = g(i-j)$$

or

$$(v_6 - 2c_6) - (v_3 - 2c_3) = \Delta t\, g(i-j) \tag{14}$$

Equations (13) and (14) can be solved simultaneously to give the conditions at point 6.

Adding equations (13) and (14) gives

$$v_6 = \frac{v_5 + v_3}{2} + (c_5 - c_3) + \Delta t\, g(i-j)$$

subtracting gives

$$c_6 = \frac{v_5 - v_3}{4} + \frac{c_5 + c_3}{2}$$

i.e. the solution at point 6 has been found.

3.6 *The domain of dependence and zone of influence*

The above concepts of which points influence other lead to two important ideas that is the **domain of dependence** and **zone of influence**. These are shown diagrammatically below.

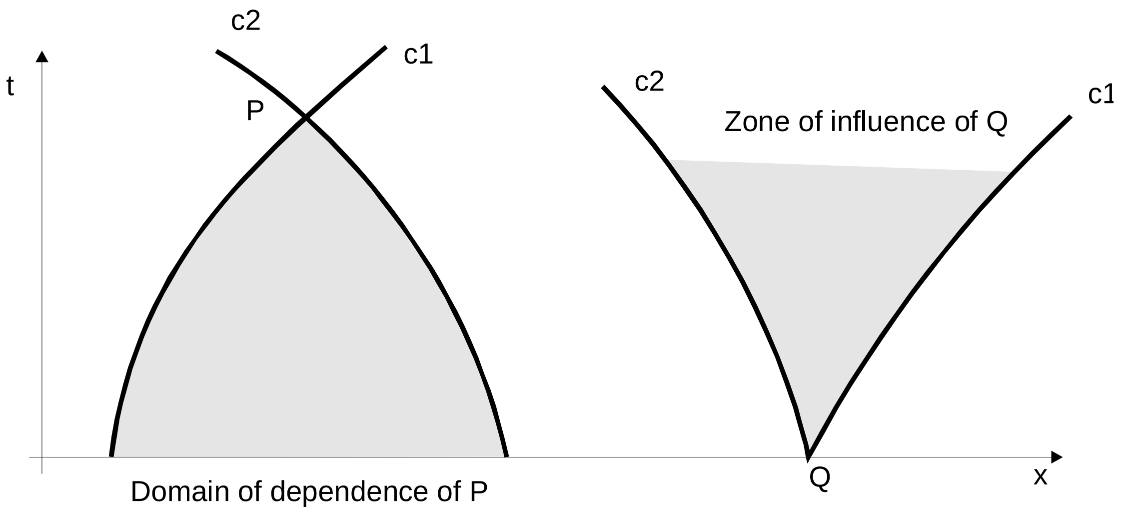

Figure 3, Domain of dependence and zone of influence.

Conditions at point P are determined solely by the conditions bounded within the two characteristics C1 and C2. Likewise the conditions at point Q determine the conditions within the area bounded by the characteristic lines

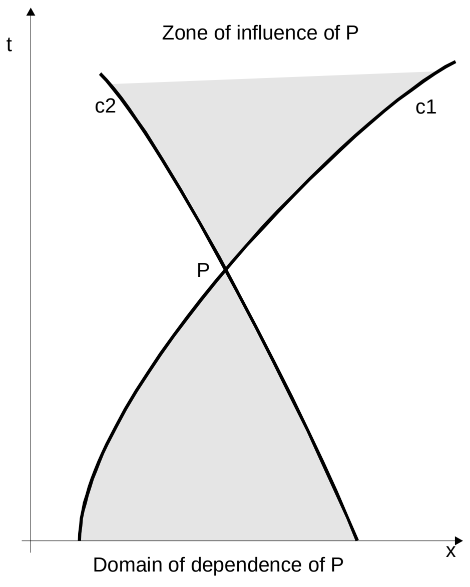

Figure 4

The figure left shows the same regions when the characteristic lines are shown extended through the point.

The significance to numerical methods and stability, is that the numerical method must take only information from within the domain of dependence of P when determining the value of P. In practise this limits the size of time step possible – hence solutions usually take longer to compute than desired.

3.7 *A first numerical method of solution*

Two commonly used numerical methods will be demonstrated in this note. One is based on a staggered grid of solution point and doesn’t actually use the characteristic equations and the second on a rectangular grid which does make explicit use of these. Both methods are referred to as the Method of Characteristics – although they are both really just approximations. They do use the ideas of where information if travelling from - which is itself determined by the characteristics.

3.7.1 The Staggered Grid Method

Consider the grid shown in Figure 5, in which the nodes are spaced at regular *Δx* intervals along the direction of the *x* axis and at *Δt* intervals along the *t* axis, the nodes are staggered a distance *Δx/2* between t and *t+Δt* . (Note that the Δt spacing may not be equal - more about that below.)

We wish to calculate the solutions at point P which is at time *t+Δt*, we know **all** the solutions values at the previous time, at time *t*.

The two points to the left and right of P, but at the previous time level, *t*, are points L and M respectively. Drawing two characteristics through these gives the arrangement shown in Figure 5. The partial derivatives of the dependent variables can be written

$$\frac{\partial v}{\partial x} = \frac{v_R - v_L}{\Delta x} \qquad \frac{\partial c}{\partial x} = \frac{c_R - c_L}{\Delta x}$$

and

$$\frac{\partial v}{\partial t} = \frac{v_P - v_M}{\Delta t} \qquad \frac{\partial c}{\partial t} = \frac{c_P - c_M}{\Delta t}$$

where point M is mid way between L and R and mean values can be used.

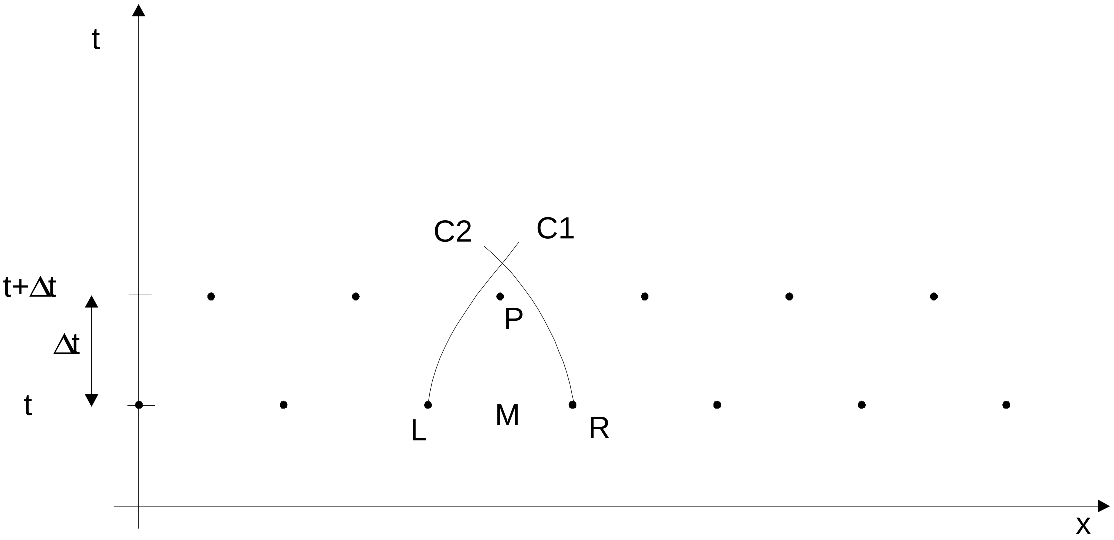

Figure 5, A staggered grid

Substituting these into our rearranged St Venant equation (3) gives

$$2c_M \frac{(c_R - c_L)}{\Delta x} + v_M \frac{(v_R - v_L)}{\Delta x} + \frac{(v_P - v_M)}{\Delta t} = g(i - j) \tag{15}$$

This equation is explicit in v_P i.e. we can calculate v_P directly

$$v_P = v_M + \frac{\Delta t}{\Delta x}\left[2c_M (c_L - c_R) + v_M (v_L - v_R) + g\Delta x (i - j)_M\right]$$

And similarly for equation (4)

$$c_M \frac{(v_R - v_L)}{\Delta x} + 2v_M \frac{(c_R - c_L)}{\Delta x} + 2\frac{(c_P - c_M)}{\Delta t} = 0 \tag{16}$$

which is explicit in c_p giving

$$c_P = c_M + \frac{1}{2}\frac{\Delta t}{\Delta x}\left[2v_M\left(c_L - c_R\right) + c_M\left(v_L - v_R\right)\right]$$

These expressions for v_p and c_p enable the solution to be determined since all variable on the right hand side are from the known conditions at the starting time level *t*. they can be used to calculate the solutions at every advanced time point on the $t+\Delta t$ level by moving from point to point using each ones lower left and right point values.

This statement needs qualifying since the expressions are not true for the upstream and downstream points (when x=0 and x = L). At these boundaries we must derive special equations - known as **boundary conditions** to describe exactly what changes are occurring there. The question of boundary conditions will be addressed later in this note.

3.8 Stability and Accuracy

The method described above is clearly quite simple to implement - although there are a lot of repetitive calculations involved if we wish to calculate for every point on a channel. A computer makes it particularly straightforward to calculate all the point and advance (or march) through time updating all values at every point and at every time level. However one must be extremely careful that the solution remains **stable** and **accurate**.

Stability and accuracy are controlled by the values Δt and Δx. We can determine a stability criterion by considering how information is being passed in our solution - or as we have seen by looking at the domain of dependence to see where the information at any point comes from.

Consider Figure 6 which shows the characteristics around the point P from our earlier calculation (see Figure 5)

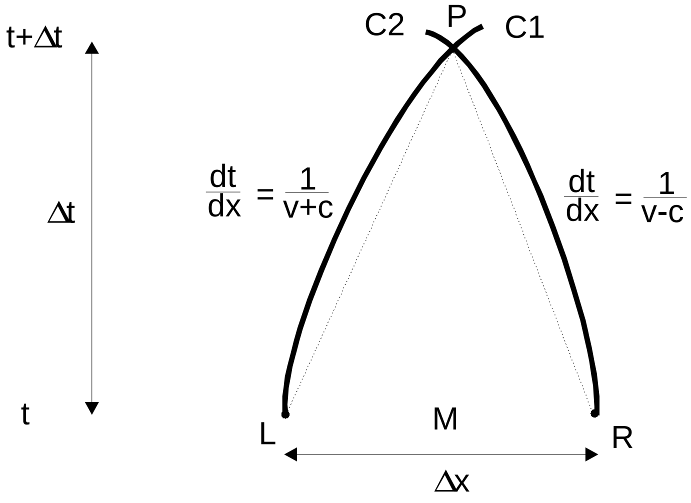

Figure 6, Characteristics around P

The forward characteristic through L has slope $\frac{dt}{dx} = \frac{1}{v+c}$ and the negative characteristic through R has slope $\frac{dt}{dx} = \frac{1}{v-c}$. From our earlier arguments point P can **only** receive information from points L and R if it lies within the domain of dependence of L and R - i.e. in the region bounded by the two characteristics.

Therefore to remain within this domain of dependence this criteria for the forward characteristic must be satisfied

$$\frac{dx}{dt} > (v + c) \tag{17}$$

This stability condition is often referred to as the **Courant** or **CFL** (Courant-Friedrichs-Lewy) condition

and since $dx = \Delta x/2$ for our staggered grid the stability criteria is

$$\frac{\Delta x}{\Delta t} > 2(v + c)$$

Considering the backward characteristic we get the criteria for stability

$$\frac{dx}{dt} > (v - c)$$

but this is unnecessary as it is already taken account of by criteria (17) (as $(v+c) > (v-c)$).

Clearly to apply (17) values of v and p must be known. But which do we choose? The values at L, R and P are all different (and those at P are not known beforehand). In practice the values at the known level must be used and some factor (≈ 0.9) introduced to take into account that the values at P may be so different to those at L or R as to change the gradient of the characteristics significantly.

To actually use the stability condition in a calculation it is usual to choose a fixed grid spacing, Δx, then determine the maximum value of $(v+c)$ for each point on the know time level then calculating a Δt. So for the staggered grid the time step would be

$$\Delta t = 0.9 \frac{\Delta x}{2(v + c)_{\max}} \tag{18}$$

This ensures stability of calculation for every point.

For accuracy the stability criteria must be obeyed and ideally the lines joining L and P and R and P should be as close to the characteristics as possible. This is sometimes not possible if for example there are greatly differing velocities along a channel when equation (18) would force a very low time step at a point where is it not required.

3.9 A second numerical solution method

While the first solution method shown used a finite difference solution to the actual St Venant equations, this second method takes the characteristic form of the equations and solves these along the characteristic paths. The solution to these is the same as the solution obtained from the original St Venant equations.

3.9.1 The method of characteristics on a rectangular grid

For this method we choose a regularly spaced grid arrangement as shown in Figure 7.

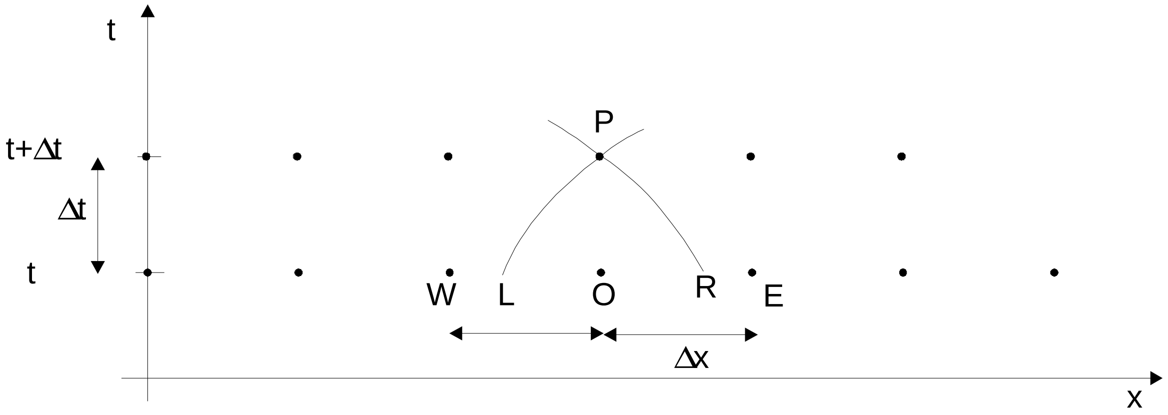

Figure7 - A rectangular grid with characteristics through P.

This figure shows the advanced time solution point P, with the forward and backward characteristics passing through it. These intersect with the initial time level line at points L (left) and R (right) respectively.

Recall the characteristic equations:

$$\frac{d(v+2c)}{dt} = g(i-j) \qquad \text{for} \qquad \frac{dx}{dt} = (v+c) \tag{19}$$

and

$$\frac{d(v-2c)}{dt} = g(i-j) \qquad \text{for} \qquad \frac{dx}{dt} = (v-c) \tag{20}$$

We will apply these along the forward and backward characteristics. First we must simplify the situation by assuming that the characteristics are straight lines. This is true only if nothing is changing (steady uniform flow) - a situation we are not very interested in. However if the time step is small the assumption usually gives very good solutions. Figure 8 shows a close up of the grid and straight characteristics associated with point P.

Secondly we must construct the characteristics. We know the slope when the forward characteristic crosses the known level, *t*, it is given by $\frac{dx}{dt} = (v_L + c_L)$ and the slope when it passes through P is given by $\frac{dx}{dt} = (v_P + c_P)$. The problem is that we don't know the values at either L or P! We must make an approximation. The usual approximation to determine the slope of the characteristics is to use the point at the same *x* position as P, but on the known time level i.e. the values at point O. The time step is Δt so the slope of the forward characteristic is

$$\frac{\Delta x_L}{\Delta t} = (v_O + c_O)$$

and the backward characteristic is

$$\frac{\Delta x_R}{\Delta t} = (v_O - c_O)$$

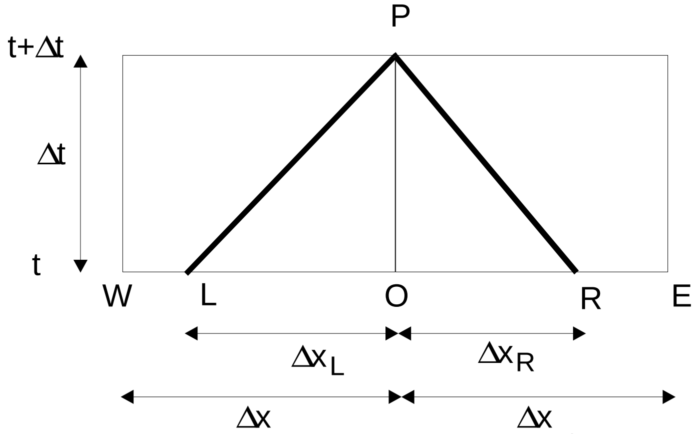

Figure 8 Grid and associated characteristics for point P

Now we can apply equation (19) along the forward characteristic, L-P, to give:

$$(v_P + 2c_P) - (v_L + 2c_L) = \Delta t\, g(i - j)_L \tag{21}$$

And equation (20) along the backward characteristic, R-P, to give:

$$(v_P - 2c_P) - (v_R - 2c_R) = \Delta t\, g(i - j)_R \tag{22}$$

Solving equations (21) and (22) simultaneously gives the conditions at point P.

Adding equations (21) and (22) gives

$$v_P = \frac{v_L + v_R}{2} + (c_L - c_R) + \frac{\Delta t\, g}{2}\left[(i - j)_L + (i - j)_R\right] \tag{23}$$

subtracting gives

$$c_P = \frac{v_L - v_R}{4} + \frac{c_L + c_R}{2} + \frac{\Delta t\, g}{4}\left[(i - j)_L - (i - j)_R\right] \tag{24}$$

These are the solution at point P.

To simplify matters we might have chosen to calculate the friction slope at the mid point O (as this is at the same *x* position as P) which would give for the forward characteristic

$$(v_P + 2c_P) - (v_L + 2c_L) = \Delta t\, g(i - j)_O \tag{25}$$

for the backward characteristic

$$(v_P - 2c_P) - (v_R - 2c_R) = \Delta t\, g(i - j)_O \tag{26}$$

and the solution is

$$v_P = \frac{v_L + v_R}{2} + (c_L - c_R) + \Delta t\, g(i - j)_O$$

(27)

$$c_P = \frac{v_L - v_R}{4} + \frac{c_L + c_R}{2}$$

(24)

To calculate this solution we must obtain the values at the point L and R. They are on the initial time level where we know all the solution, but we only know the solution at the node points (W, O and E).If we know the position of L and R we can interpolate between these know values to get v_L, c_L, v_R, and c_R. they are not at the node

A simple linear interpolation procedure is all that is necessary i.e. for v_L

$$v_L = v_O - \frac{\Delta x_L}{\Delta x}(v_O - v_W)$$

and for v_R

$$v_R = v_O - \frac{\Delta x_R}{\Delta x}(v_O - v_E)$$

3.9.2 The stability criteria

The stability criteria discussed in section 3.8 and shown in equation (17)

$$\frac{dx}{dt} > (v + c)$$

is valid for the method of characteristics on a rectangular grid. In this case, $dt = \Delta t$ and $dx = \Delta x$ so the limit on the time step is

$$\Delta t < \frac{\Delta x}{(v + c)}.$$

This allows a time step **twice** that of the staggered grid method.

To implement this, as with the staggered grid method the maximum value of *(v+c)* for each point on the know time level should be determined then a Δt calculated. Using a suitable *safety factor* the time step would be

$$\Delta t = 0.9 \frac{\Delta x}{(v + c)_{\max}}$$

Choosing a time step like this ensures that the Δx_L and Δx_R are both less than Δx.

3.10 Boundary conditions

At any boundary there is only one characteristic present because the other is outside the channel. For an upstream boundary we have the backward characteristic, for the downstream boundary we have the forward characteristic - as shown in Figure 9.

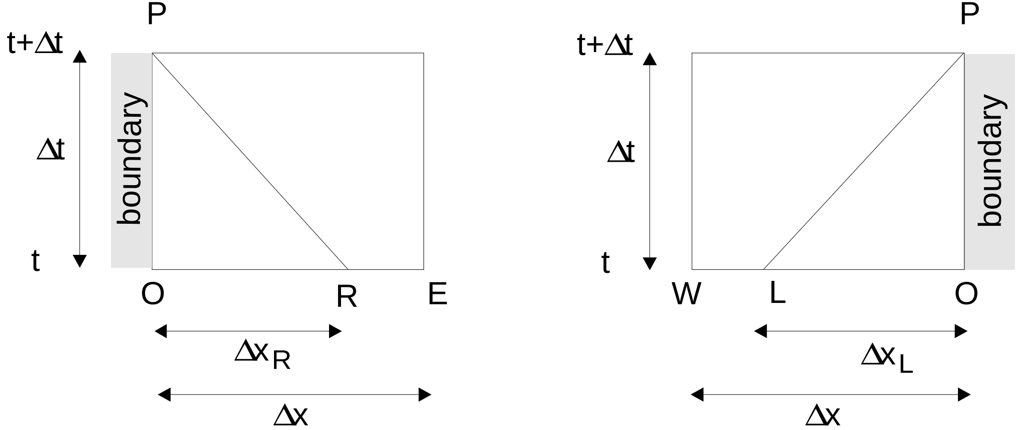

Figure 9 Characteristics at upstream and downstream boundaries

Instead of the second characteristic equation we have a **boundary condition equation**. This will either specify the depth (or stage) at P - e.g. from a stage curve possibly from historical records. Or it will specify the discharge at P - e.g. from a flood hydrograph.

3.10.1 Depth specified - Upstream boundary

This is very simple to solve. The values of depth h_P is given so c_P readily calculated (from $c = \sqrt{gh}$) then the only solution for v_P is required.

From the backward characteristic equation

$$(v_P - 2c_P) - (v_R - 2c_R) = \Delta t\, g(i - j)_R$$

so

$$v_P = v_R - 2(c_R - c_P) + \Delta t\, g(i - j)_R$$

3.10.2 Depth specified - Downstream boundary

This is again very simple to solve. The values of depth h_P is given so c_P can be calculated and only the solution for v_P is required.

From the forward characteristic equation

$$(v_P + 2c_P) - (v_L + 2c_L) = \Delta t\, g(i - j)_L$$

so

$$v_P = v_L + 2(c_L - c_P) + \Delta t\, g(i - j)_L$$

3.10.3 Discharge specified - Upstream boundary

In this case the discharge Q_p is given, but this must be converted to a value in terms of v_p or c_p, so the shape of the channel section must be known to allow the area of flow to be calculated from depth. For simplicity we will assume a rectangular channel so

$$\begin{aligned} Q_P &= Av_P \\ &= bh_P v_P \\ &= b\frac{c_P^2}{g}v_P \end{aligned}$$

or

$$v_P = \frac{Q_P g}{bc_P^2}$$

Substitute this in to the backward characteristic gives

$$\left(\frac{Q_P g}{bc_P^2} - 2c_P\right) - \left(v_R - 2c_R\right) = \Delta t\, g\left(i - j\right)_R$$

which can be rearranged to give

$$2c_P^3 + c_P^2\left(v_R + 2c_R + \Delta t\, g\left(i - j\right)_R\right) - \frac{Q_P g}{b} = 0$$

This is usually solved using some iteration technique - e.g. Newton-Raphson.

3.10.4 Discharge specified - Downstream boundary

This is very similar to the upstream boundary except in this case we use the forward characteristic:

$$\left(\frac{Q_P g}{bc_P^2} + 2c_P\right) - \left(v_L + 2c_L\right) = \Delta t\, g\left(i - j\right)_L$$

which can be rearranged to give

$$2c_P^3 - c_P^2\left(v_L + 2c_L - \Delta t\, g\left(i - j\right)_L\right) + \frac{Q_P g}{b} = 0$$

And again, this can be solved using some iteration technique - e.g. Newton-Raphson.

3.11 Super-Critical flow

When the flow is supercritical the wave speed, *c*, is greater than the velocity of flow. Using the equation for the backward characteristic to calculate Δx_L we get

$$\Delta x_L = \Delta t\left(v_L - c_L\right)$$

but $c_L > v_L$ so Δx_L is negative. The characteristic around point P then look like those in Figure 10

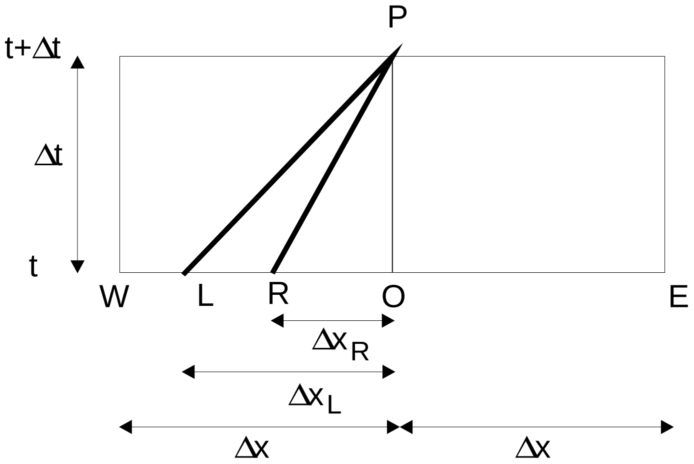

Figure 10: Characteristic in super-critical flow

The backward characteristic has now moved over to the left side and has a positive slope. It will never move so far as to be to the left of the forward characteristic - but in very fast flow they can be very close.

The solution method is exactly the same as that set out in section 3.9. The only change is that v_R and c_R are now interpolated between points W and O (instead of E and O).

3.11.1 Boundary conditions in super-critical flow

At the upstream boundary in super-critical flow **both** the characteristics are outside the channel. In which case two boundary condition equations must be given to specify both *v* and *c* at point P.

At the downstream boundary there are two characteristics available, as shown in Figure 11, and no boundary condition is required.

(It is often a common error that a boundary condition is applied to a downstream boundary with super-critical flow. In this situation you have three equations for two unknowns and the problem over specified - and usually an error manifesting as instabilities will result.)

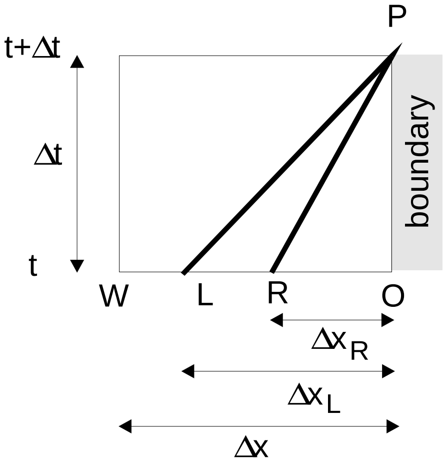

Figure 11: Characteristics at a super-critical downstream boundary

The solution method for the supercritical downstream boundary is the same as the solution for either sub or super-critical flow in the main stream (see section 3.9).